4/1/26

Return HW

<u>Def</u> Vectors $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$ are called <u>orthonormal</u> if

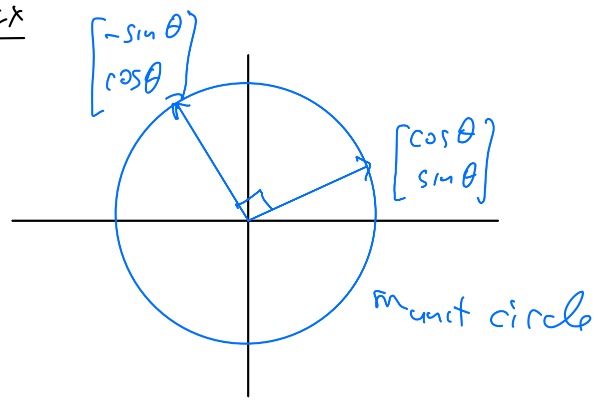
$\underbrace{\|\vec{u}_i\| = 1}_{\text{"normal"}}$ for all i <u>and</u> $\underbrace{(\vec{u}_i \cdot \vec{u}_j) = 0}_{\text{orthogonal}}$ $\forall\, i \neq j$

So $\vec{u}_1, \ldots, \vec{u}_m$ are orthonormal if

$$(\vec{u}_i \cdot \vec{u}_j) = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

<u>Ex</u> standard basis

<u>Ex</u>



<u>Th</u> (5.1.3)

Let $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$, none $= \vec{0}$.

If they are orthogonal, then they are lin. indep.
(e.g. if they're orthonormal)

<u>pf</u>

(a) Say $c_1\vec{u}_1 + \cdots + c_m\vec{u}_m = \vec{0}$. want to show $c_1 = \cdots = c_m = 0$

Take dot prod with $\vec{u}_1$

$$0 = (c_1\vec{u}_1 + \cdots + c_n\vec{u}_m)\cdot\vec{u}_1 = c_1(\vec{u}_1\cdot\vec{u}_1) + c_2\underbrace{(\vec{u}_2\cdot\vec{u}_1)}_{=0} + \cdots + c_m\underbrace{(\vec{u}_m\cdot\vec{u}_1)}_{=0}$$

$$= c_1\underbrace{(\|\vec{u}_1\|^2)}_{\neq 0}$$

So $c_1 = 0$

Similarly, $c_2 = \cdots = c_m = 0 \implies \vec{u}_1, \ldots, \vec{u}_m$ are lin. indep.

Def Let $\vec{u}_1, \ldots, \vec{u}_n$ be vectors in $\mathbb{R}^n$. If they are orthonormal then they form an orthonormal basis for $\mathbb{R}^n$.

(Note orthogonal already implies lin. indep, and the fact that there are n of them means they form a basis.)

Orthogonal projections, part II

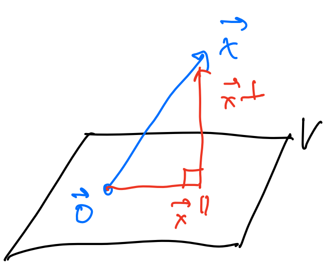
We talked about this in $\mathbb{R}^2$. Now we do the general case.

Th (5.1.4)

Let $\vec{x} \in \mathbb{R}^n$ and let V be a subspace of $\mathbb{R}^n$. Then we can write

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

where $\vec{x}^{\parallel} \in V$ and $\vec{x}^{\perp}$ is perpendicular (= orthogonal) to V (i.e. to every element of V).



($\vec{x}^{\parallel}$ is on V, $\vec{x}^{\perp}$ is perp. to V)

(cont.)

The vector $\vec{x}^{\parallel}$ is the orthogonal projection of $\vec{x}$ to V, denoted by $\text{proj}_V(\vec{x})$.
The function $\vec{x} \longmapsto \text{proj}_V(\vec{x})$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a linear transf.

proof 1st 2 comments

- We will see shortly that given any subspace V of $\mathbb{R}^n$, it has an orthonormal basis. We'll accept this for now.
- In terms of an orthonormal basis for V, this proof will not only prove that $\text{proj}_V(\vec{x})$ makes sense, it will give us a formula for it.

Let $\{\vec{u}_1, \dots, \vec{u}_m\}$ be an orthonormal basis for V. We accept that this exists.

Idea:

- We don't know yet how to find $\text{proj}_V(\vec{x})$. That's part of the proof.
- Whatever it turns out to be, it's in V so it's a linear comb

$$\vec{x}^{\parallel} = \text{proj}_V(\vec{x}) = c_1\vec{u}_1 + \dots + c_m\vec{u}_m$$

- We'll figure out what $c_1, \dots, c_m$ have to be, and use this to define $\text{proj}_V(\vec{x})$.
- For simplicity of notation we'll write $\vec{x}^{\parallel}$ instead of $\text{proj}_V(\vec{x})$ for the proof.

So we need

$\vec{x}^{\parallel} \in V$

$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$

key $\rightarrow$ $\vec{x}^{\perp}$ is orthogonal to $\vec{u}_1, \dots, \vec{u}_m$ (hence to every element of V)

So $\vec{x}^{\parallel} = c_1 \vec{u}_1 + \cdots + c_m \vec{u}_m$. This automatically means $\vec{x}^{\parallel} \in V$.

$\vec{x}^{\perp} = \vec{x} - c_1 \vec{u}_1 - \cdots - c_m \vec{u}_m$ this automatically gives $\vec{x} = \vec{x}^{\perp} + \vec{x}^{\parallel}$

<u>Claim</u> We must have

$$c_1 = (\vec{u}_1 \cdot \vec{x})$$
$$c_2 = (\vec{u}_2 \cdot \vec{x})$$
$$\vdots$$
$$c_m = (\vec{u}_m \cdot \vec{x})$$

So

$$\vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m \qquad \left(= \mathrm{proj}_V(\vec{x}) \right)$$

and $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$

We have to show that with this choice of coefficients, $\vec{x}^{\perp}$ is orthogonal to V, ie to each $\vec{u}_i$

Do $\vec{u}_1$. The others are identical

$$\vec{u}_1 \cdot \vec{x}^{\perp} = \vec{u}_1 \cdot \left(\vec{x} - \vec{x}^{\parallel} \right)$$

$$= \vec{u}_1 \cdot \left(\vec{x} - (\vec{u}_1 \cdot \vec{x})\vec{u}_1 - (\vec{u}_2 \cdot \vec{x})\vec{u}_2 - \cdots - (\vec{u}_m \cdot \vec{x})\vec{u}_m \right)$$

$$= (\vec{u}_1 \cdot \vec{x}) - (\vec{u}_1 \cdot \vec{x})(\vec{u}_1 \cdot \vec{u}_1) - (\vec{u}_2 \cdot \vec{x})(\vec{u}_1 \cdot \vec{u}_2) - \cdots - (\vec{u}_m \cdot \vec{x})(\vec{u}_1 \cdot \vec{u}_m)$$

$$= (\vec{u}_1 \cdot \vec{x}) - (\vec{u}_1 \cdot \vec{x})(1) - 0 - \cdots - 0$$

$$= 0$$

So $\vec{x}^{\perp}$ is orthogonal to $\vec{u}_1$, and similarly to every other $\vec{u}_i$, hence to every element of V.

So we proved the key step.

Leave to you to check linearity (use the formula in the claim.) //

<u>Def</u> $\mathrm{proj}_V(\vec{x}) = \vec{x}^{\parallel}$

$$= (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

for all $\vec{x} \in \mathbb{R}^n$. (This is the formula in terms of $\vec{u}_1, \ldots, \vec{u}_m$.)